Per Aspera Ad Astra

Candidate Surname	Other Names

A-Level Further Mathematics Core Pure 1 Paper Reference: PAAA-01 Made by: Gavin Arora	Time allowed: **2 hours**
	Total Marks:

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for algebraic manipulation, calculus, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer all questions in the spaces provided.
- You should show sufficient working to make your method clear.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- There are **8 questions** in this question paper. The total mark is **75**.
- The marks for each question are shown in brackets.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Seek out **professional therapy** after attempting this question paper.

Q1.

(a) Given that

$$f(x) = 4x - 8x^3 + \frac{8}{3}x^5 - \cdots$$

is the result of a Maclaurin series expansion, show that

$$\int f(x)\,dx = Ax \sin 2x + \cos 2x + C,$$

where C is an arbitrary constant and A is a constant to be found.

(6)

(b) Hence, find the mean value of $f(x)$ over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, leaving your answer in exact form.

(2)

(Total for question = 8 marks)

Q2.

Evaluate the following, giving your answer in its simplest form:

$$\int \left(\sum_{r=1}^{x} \log_e \left(\frac{r^2 + 2r - 3}{r^2 - 1} \right) \right) dx$$

(6)

(Total for question = 6 marks)

Q3.

a) Prove that

$$\sinh(x + iy) \equiv \cos(y)\sinh(x) + i\sin(y)\cosh(x)$$

(3)

b) Hence, determine the **general** solution of the following equation:

$$\sinh(x + iy) = e^{\frac{1}{3}\pi i},$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

(10)

(Total for question = 13 marks)

Q4.

In this question you must show detailed reasoning.

Find a vector $\boldsymbol{v}$ which has the following properties:

- It is a unit vector.
- It is parallel to the plane $2x + 2y + z = 10$.
- It makes an angle of $45°$ with the normal to the plane $x + z = 5$.

(8)

(Total for question = 8 marks)

Q5.

Three interacting populations live in a large pond:

- $x(t)$ represent algae,
- $y(t)$ represent small fish,
- $z(t)$ represent frogs,

where t is measured in months, and the three populations are measured in hundreds.

The populations are modelled by the following differential questions:

$$\frac{dx}{dt} = 2 - x, \quad \frac{dy}{dt} = x - 2y + z, \quad \frac{dz}{dt} = -z,$$

where

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = 5.$$

a) Interpret the meaning of each of the terms:
  i) $2 - x$ in the first equation,
  ii) $-z$ from the third equation.

(2)

b) Find a particular solution for the number of:
  i) algae
  ii) frogs

at t months.

(6)

c) Hence, show that $y(t) = 1 + 3e^{-t} - 3e^{-2t}$.

(4)

d) Using your three solutions, determine whether the three populations are ever equal at any point in time.

(2)

e) By considering the long-term behaviour of each population, comment on the reliability of the model, giving a reason for your answer.

(2)

(Total for question = 16 marks)

Q6.

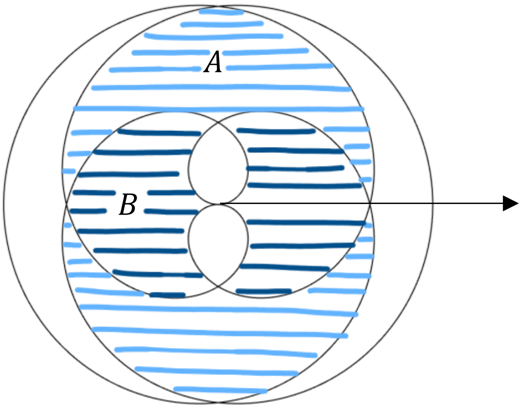

Figure 1

Figure 1 shows a plan view of a 2-tier swimming pool. The plan view, which is scaled down by a factor of 10 from the real pool, is defined by the polar curve with equation

$$r = \sin\frac{\theta}{4} \qquad 0 \leq \theta \leq 8\pi$$

Region A represents the upper tier of the swimming pool. This tier has a depth of 2 m, measured from ground level.

Region B represents the lower tier of the swimming pool. This tier has a depth of x m, measured from the top surface of upper tier of the pool.

The unshaded parts of the plan view represent layers of stone tiling at ground level.

The two tiers are separated by a border of negligible width, so that there is no leakage of water from the upper tier into the lower tier.

a) Given that the maximum volume of water that can be held in the lower tier is half of the maximum volume of water that can be held in the upper tier, find the value of x.

(8)

b) State a limitation of the model, and describe how this would affect your answer to part (a).

(2)

(Total for question = 10 marks)

Q7.

In this question you must show detailed reasoning.

The complex number z satisfies the relationship

$$\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$$

Show that the locus of z is a circular arc, stating...

- ... the coordinates of its endpoints.
- ... the coordinates of its centre.
- ... the length of its radius.

(8)

(Total for question = 8 marks)

Q8.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n = n\mathbf{A} - (n-1)\mathbf{I}$$

where $n \geq 1, n \in \mathbb{N}$, and $\mathbf{I}$ represents the identity matrix.

(6)

(Total for question = 6 marks)

END OF QUESTION PAPER.