Question	Scheme	Marks
1(a)	$f(x) = 4x\left(1 - 2x^2 + \frac{2}{3}x^4 + \cdots\right)$	B1
	$= 4x\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots\right)$	M1
	$= 4x\cos(2x)$	A1
	$\therefore \int f(x)\,dx = \int 4x\cos(2x)\,dx$	M1
	$= 4x\left(\frac{1}{2}\sin(2x)\right) - 4\left(-\frac{1}{4}\cos(2x)\right) + C = 2x\sin(2x) + \cos(2x) + C$	M1A1
Alt. 1(a)	$f(x) = A\sin(2x) + 2Ax\cos(2x) - 2\sin(2x) = (A-2)\sin(2x) + 2Ax\cos(2x)$	B1
	$= (A-2)\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots\right) + 2Ax\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots\right)$ $= x(4A-4) + x^3\left(-\frac{16}{3}A + \frac{8}{3}\right) + x^5\left(\frac{8}{5}A - \frac{8}{15}\right) + \cdots$	M1
	$x(4A-4) = 4x \Rightarrow A = 2 \Rightarrow f(x) = 2(2)x\cos(2x) = 4x\cos(2x)$	M1A1
	Checks for consistency: $A = 2 \Rightarrow x^3\left(-\frac{16}{3}(2) + \frac{8}{3}\right) = -8x^3$ $A = 2 \Rightarrow x^5\left(\frac{8}{5}(2) - \frac{8}{15}\right) = \frac{8}{3}x^5$ $\therefore f(x) = 4x\cos(2x)$ is consistent with the given Maclaurin series expansion $\therefore \int f(x)\,dx = 2x\sin(2x) + \cos(2x) + C$	dM1A1
		(6)
1(b)	$\frac{1}{\frac{\pi}{2} - \frac{\pi}{4}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)\,dx = \frac{4}{\pi}\left[2x\sin(2x) + \cos(2x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$	M1
	$= \frac{4}{\pi}\left(\pi(0) - 1 - \frac{\pi}{2}(1) - 0\right) = -\frac{4}{\pi}\left(\frac{\pi}{2} + 1\right)$ o.e.	A1

		(2)
		(5 marks)

Question	Scheme	Marks
2	$\sum_{r=1}^{x} \ln\left(\frac{(r+3)\cancel{(r+1)}}{\cancel{(r+1)}(r-1)}\right) = \sum_{r=1}^{x} \ln(r+3) - \ln(r+1)$	B1
	$= \cancel{\ln(4)} - \ln(2)$ $+ \cancel{\ln(5)} - \ln(3)$ $+ \cancel{\ln(6)} - \cancel{\ln(4)}$ $+ \cdots$ $+ \cancel{\ln(x+1)} - \cancel{\ln(x-1)}$ $+ \ln(x+2) - \cancel{\ln(x)}$ $+ \ln(x+3) - \cancel{\ln(x+1)}$ $= \ln(x+2) + \ln(x+3) - \ln\,(6)$	M1A1
	$\int (\ln(x+2) + \ln(x+3) - \ln(6))\,dx$	M1
	$= (x+a)\ln(x+a) + (x+b)\ln(x+b) - 2x - x\ln(6)\ [+C]$	B1
	$= (x+2)\ln(x+2) + (x+3)\ln(x+3) - 2x - x\ln(6) + C$ o.e.	A1
		(6)
		(6 marks)

Question	Scheme	Marks
3(a)	$\sinh(x+iy)=\dfrac{e^{x+iy}-e^{-(x+iy)}}{2}$	B1
	$=\dfrac{e^x[\cos(y)+i\sin(y)]-e^{-x}[\cos(y)-i\sin(y)]}{2}$	M1
	$=\cos(y)\left(\dfrac{e^x-e^{-x}}{2}\right)+i\sin(y)\left(\dfrac{e^x+e^{-x}}{2}\right)$ $=\cos(y)\sinh(x)+i\sin(y)\cosh(x)$	A1
		(3)
3(b)	$e^{\frac{\pi}{3}i}=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\Rightarrow\cos(y)\sinh(x)=\cdots$	M1
	$\cos(y)\sinh(x)=\dfrac{1}{2},\sin(y)\cosh(x)=\dfrac{\sqrt{3}}{2}$	A1
	$\therefore\sinh(x)=\dfrac{1}{2\cos(y)}\ \ \mathit{and}\ \ \cosh(x)=\dfrac{\sqrt{3}}{2\sin(y)}$	B1
	$\cosh^2(x)-\sinh^2(x)=1\Rightarrow\dfrac{3}{4\sin^2(y)}-\dfrac{1}{4\cos^2(y)}=1$ $\therefore 3\cos^2(y)-\sin^2(y)=4\sin^2(y)\cos^2(y)$	M1
	$\therefore 3(1-\sin^2(y))-\sin^2(y)=4\sin^2(y)(1-\sin^2(y))$ $\therefore 4\sin^4(y)-8\sin^2(y)+3=0$ $(2\sin^2(y)-3)(2\sin^2(y)-1)=0\Rightarrow\sin(y)=\cdots$	M1
	$\sin(y)\neq\pm\dfrac{\sqrt{6}}{2}$ or $\sin(y)=\pm\dfrac{\sqrt{2}}{2}$	A1
	Since $\cosh(x)>0,\sin(y)>0\Rightarrow\sin(y)=\dfrac{\sqrt{2}}{2}$	B1
	$\therefore y=\dfrac{\pi}{4}+2\pi k$ or $y=\dfrac{3\pi}{4}+2\pi k,\forall k\in\mathbb{Z}^+$	A1

	$\frac{\sqrt{2}}{2}\cosh(x) = \frac{\sqrt{3}}{2} \Rightarrow \cosh(x) = \frac{\sqrt{6}}{2}$	M1
	$\therefore x = \cosh^{-1}\left(\frac{\sqrt{6}}{2}\right)$ $= \ln\left(\frac{\sqrt{6}}{2} + \sqrt{\left(\frac{\sqrt{6}}{2}\right)^2 - 1}\right)$ $= \ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)$ o.e.	A1
		(10)
		(13 marks)

Question	Scheme	Marks
4	$Let\ \boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$	
	$\lvert\boldsymbol{v}\rvert = 1 \Rightarrow a^2 + b^2 + c^2 = 1$	B1
	$\boldsymbol{v} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0 \Rightarrow 2a + 2b + c = 0$	B1
	$\boldsymbol{v} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \lvert\boldsymbol{v}\rvert \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \cos(45) \Rightarrow a + c = 1 \Rightarrow a = 1 - c$	B1
	$\therefore (1-c)^2 + b^2 + c^2 = 1 \Rightarrow 2c^2 - 2c + b^2 = 0$	M1
	$2(1-c) + 2b + c = 0 \Rightarrow b = \frac{1}{2}(c-2)$	M1
	$\therefore 2c^2 - 2c + \left(\frac{1}{2}(c-2)\right)^2 = 0$ $\therefore 2c^2 - 2c + \frac{1}{4}(c^2 - 4c + 4) = 0$ $\therefore 9c^2 - 12c + 4 = 0 \Rightarrow (3c-2)^2 = 0 \Rightarrow c = \cdots$	M1
	$c = \frac{2}{3} \Rightarrow a = \frac{1}{3}$ and $b = -\frac{2}{3}$	A1
	$\therefore \boldsymbol{v} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ o. e.	A1
		(8)
		(8 marks)

Question	Scheme	Marks
5(a)(i)	Represents **net growth rate** of algae. Where the constant **2** suggests **continous/constant birth rate** ... While $-$ **x** indicates a **death rate proportional to current population.**	A1
(a)(ii)	Frog population **declining** at a rate **proportional to current population.**	A1
		(2)
5(b)(i)	$\int \frac{1}{2-x} dx = \int dt \Rightarrow -\ln(2-x) = t + c$	B1
	Using $x(0) = 0 \Rightarrow -\ln(2) = c \Rightarrow -\ln(2-x) = t - \ln(2)$	M1
	$\therefore \frac{2}{2-x} = e^t \Rightarrow 2 - x = 2e^{-t} \Rightarrow x = 2 - 2e^{-t}$	A1
5(b)(ii)	$\int \frac{1}{z} dz = -\int dt \Rightarrow \ln(z) = -t + c$	B1
	Using $z(0) = 5 \Rightarrow \ln(5) = c \Rightarrow \ln(z) = -t + \ln(5)$	M1
	$\therefore t = \ln\left(\frac{5}{z}\right) \Rightarrow e^t = \frac{5}{z} \Rightarrow z = 5e^{-t}$	A1
		(6)
5(c)	$\frac{dy}{dt} = 2 - 2e^{-t} - 2y + 5e^{-t}$ $\therefore \frac{dy}{dt} + 2y = 2 + 3e^{-t}$, so IF $= \cdots$	B1
	IF $= e^{\int 2dt} = e^{2t}$	M1
	$\therefore \frac{d(e^{2t}y)}{dt} = 2e^{2t} + 3e^t \Rightarrow e^{2t}y = e^{2t} + 3e^t + c \Rightarrow y(t) = 1 + 3e^{-t} + ce^{-2t}$	M1
	$y(0) = 1 \Rightarrow 1 = 1 + 3 + c \Rightarrow c = -3 \Rightarrow y(t) = 1 + 3e^{-t} - 3e^{-2t}$	A1
Alt. 5(c)	$\therefore \frac{dy}{dt} = 2 + 3e^{-t} - 2y \Rightarrow \frac{d^2y}{dt^2} = -3e^{-t} - 2\frac{dy}{dt} \Rightarrow \frac{d^2y}{dt^2} + 2\frac{dy}{dt} = -3e^{-t}$	B1

	Auxiliary equation: $m^2 + 2m = 0 \Rightarrow m(m+2) = 0 \Rightarrow m = 0, m = -2$ So C. F. is $y = A + Be^{-2t}$	M1
	P. I. is $y = \lambda e^{-t} \Rightarrow \frac{dy}{dx} = -\lambda e^{-t}, \frac{d^2y}{dt^2} = \lambda e^{-t} \Rightarrow \lambda = 3$ $\therefore$ G. S. is $y = A + Be^{-2t} + 3e^{-t}$	M1
	$y(0) = 1 \Rightarrow 1 = A + B + 3 \Rightarrow A + B = -2$ $\frac{dy}{dt} = -2Be^{-2t} - 3e^{-t}$ When $t = 0, \frac{dy}{dt} = 2 + 3(1) - 2(1) = 3 \Rightarrow \frac{dy}{dt} = -2B - 3 = 3 \Rightarrow B = -3$ $\therefore y(t) = 1 + 3e^{-t} - 3e^{-2t}$	A1
		(4)
5(d)	$5e^{-t} = 2 - 2e^{-t} \Rightarrow e^{-t} = \frac{2}{7} \Rightarrow t = 1.25 \Rightarrow z(1.25) = \frac{10}{7}$	B1
	$y(1.25) = 1 + 3e^{-1.25} - 3e^{-2(1.25)} = \frac{79}{49} \neq \frac{10}{7}$ $\therefore$ "All three populations never equal at any point in time" o. w. t. t. e.	A1
		(2)
5(e)	As $t \to \infty, x \to 2, y \to 1, z \to 0$, so model is **unrealistic** …	B1
	• As population is **unlikely to remain static**, OR • Because if there are **zero frogs**, the **whole ecosystem changes**	A1
		(2)
		(16 marks)

Question	Scheme	Marks
6(a)	$r^2 = \sin^2\left(\frac{\theta}{4}\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2}\right)$	B1
	Method to find areas of Regions A and B: $4\left[\frac{1}{2}\int_{\dots}^{\dots} r^2\, d\theta - \frac{1}{2}\int_{\dots}^{\dots} r^2\, d\theta\right]$	B1
	Region B: $4\left[\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right) d\theta - \frac{1}{2}\int_{0}^{\frac{\pi}{2}}\left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right) d\theta\right]$ $= \int_{\frac{\pi}{2}}^{\pi}\left(1 - \cos\left(\frac{\theta}{2}\right)\right) d\theta - \int_{0}^{\frac{\pi}{2}}\left(1 - \cos\left(\frac{\theta}{2}\right)\right) d\theta$	M1
	$= \left[\theta - 2\sin\left(\frac{\theta}{2}\right)\right]_{\frac{\pi}{2}}^{\pi} - \left[\theta - 2\sin\left(\frac{\theta}{2}\right)\right]_{0}^{\frac{\pi}{2}}$ $= \pi - 2(1) - \frac{\pi}{2} + \sqrt{2} - \frac{\pi}{2} + \sqrt{2}$ $= 2\sqrt{2} - 2$	A1
	Region A: $4\left[\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}}\left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right) d\theta - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right) d\theta\right]$ $= \int_{\pi}^{\frac{3\pi}{2}}\left(1 - \cos\left(\frac{\theta}{2}\right)\right) d\theta - \int_{\frac{\pi}{2}}^{\pi}\left(1 - \cos\left(\frac{\theta}{2}\right)\right) d\theta$	M1
	$= \left[\theta - 2\sin\left(\frac{\theta}{2}\right)\right]_{\pi}^{\frac{3\pi}{2}} - \left[\theta - 2\sin\left(\frac{\theta}{2}\right)\right]_{\frac{\pi}{2}}^{\pi}$ $= \frac{3\pi}{2} - \sqrt{2} - \pi + 2 - \pi + 2 + \frac{\pi}{2} - \sqrt{2}$ $= 4 - 2\sqrt{2}$	A1
	$x(2\sqrt{2} - 2) = \frac{1}{2}\left(2(4 - 2\sqrt{2})\right)$	dM1
	$\therefore x = \frac{4 - 2\sqrt{2}}{2\sqrt{2} - 2} = \sqrt{2}$	A1

		(8)
6(b)	Border is modelled as having a negligible width, which is unrealistic.	A1
	If border width is considered, answer to part (a) will be smaller OR Answer to part (a) is an overestimate	A1
		(2)
		(10 marks)

Question	Scheme	Marks
7	Let $\arg(z-2)=a$ and $\arg(z+2)=b \Rightarrow a-b=\frac{\pi}{4}$	M1
	$\therefore \tan(a)=\frac{y}{x-2}$ and $\tan(b)=\frac{y}{x+2}$	B1
	Use of $\tan(a-b)=\frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}$	M1
	$\therefore \tan\left(\frac{\pi}{4}\right)=\frac{\frac{y}{x-2}-\frac{y}{x+2}}{1+\left(\frac{y}{x-2}\right)\left(\frac{y}{x+2}\right)} \Rightarrow 1+\frac{y^2}{(x-2)(x+2)}=\frac{y}{x-2}-\frac{y}{x+2}$	M1
	$\therefore (x-2)(x+2)+y^2=y(x+2)-y(x-2)$ $\therefore x^2-4+y^2=xy+2y-xy+2y$ $\therefore x^2+(y-2)^2=8 \Rightarrow$ Locus of z is circular.	M1A1
	$\therefore$ Radius $=\sqrt{8}=2\sqrt{2}$ AND Centre at $(0,2)$	A1
	$a-b=\frac{\pi}{4}>0 \Rightarrow a>b$ Geometrically, $a>b$ only true when chords a and b are above the $x-$ axis ... $\therefore$ Locus of z is a circular **major** arc, with endpoints at $(2,0)$ and $(-2,0)$.	A1
Alt. 7	$\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$	M1
	$\therefore \arg\left(\frac{x+iy-2}{x+iy+2}\right)=\frac{\pi}{4}$	B1
	$\therefore \arg\left(\frac{[x-2]+iy}{[x+2]+iy}\right)=\frac{\pi}{4} \Rightarrow \arg\left(\frac{([x-2]+iy)([x+2]-iy)}{([x+2]+iy)([x+2]-iy)}\right)=\frac{\pi}{4}$	M1
	$\therefore \arg\left(\frac{[x^2-y^2-4]+i[4y]}{(x+2)^2+y^2}\right)=\frac{\pi}{4} \Rightarrow \frac{x^2-y^2-4}{(x+2)^2+y^2}=\frac{4y}{(x+2)^2+y^2}$	M1
	$\therefore x^2-y^2-4=4y \Rightarrow x^2+(y-2)^2=8 \Rightarrow$ Locus of z is circular	A1

	$\therefore$ Radius $= \sqrt{8} = 2\sqrt{2}$ AND Centre at $(0,2)$	A1
	Since argument is positive, $Re(z) > 0$ and $Im(z) > 0$ $\therefore x^2 + y^2 - 4 > 0 \Rightarrow x^2 + y^2 > 0$ and $4y > 0 \Rightarrow y > 0$ Geometrically, both conditions exclude the minor arc of the circle. $\therefore$ Locus of z is a circular **major** arc, with endpoints at $(2,0)$ and $(-2,0)$.	M1A1
		(8)
		(8 marks)

Question	Scheme	Marks
8	$n = 1 \Rightarrow \boldsymbol{A}^1 = \boldsymbol{A}\ and\ (1)\boldsymbol{A} - (1-1)\boldsymbol{I} = \boldsymbol{A} \Rightarrow True\ for\ n = 1$	B1
	$Assume\ true\ for\ n = k \Rightarrow \boldsymbol{A}^k = k\boldsymbol{A} - (k-1)\boldsymbol{I}$	M1
	$Inductive\ leap: n = k+1 \Rightarrow \boldsymbol{A}^{k+1} = \boldsymbol{A}^k\boldsymbol{A}^1 = [k\boldsymbol{A} - (k-1)\boldsymbol{I}]\boldsymbol{A} = \cdots$	M1
	$\left[\begin{pmatrix} k & 0 \\ 2k & k \end{pmatrix} - \begin{pmatrix} k-1 & 0 \\ 0 & k-1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 2k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 2k+2 & 1 \end{pmatrix}$	M1
	$= \begin{pmatrix} k+1-k & 0 \\ 2(k+1) & k+1-k \end{pmatrix}$ $= \begin{pmatrix} k+1 & 0 \\ 2(k+1) & k+1 \end{pmatrix} - k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $= (k+1)\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} - [(k+1)-1]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $= (k+1)\boldsymbol{A} - [(k+1)-1]\boldsymbol{I}$	A1
	$Hence, if\ true\ for\ n = k, then\ the\ statement\ is\ true\ for\ n = k+1.$ $Since\ n = 1\ is\ also\ true, the\ statement\ is\ true\ \forall\ n \in \mathbb{N}.$	A1
		(6)
		(6 marks)